

On a problem with nonperiodic frequent alternation of boundary condition imposed on fast oscillating sets

Denis I. Borisov *

Bashkir State Pedagogical University, October Revolution, St. 3a, 450000
Ufa, Russia,
E-mail: BorisovDI@ic.bashedu.ru, BorisovDI@bspu.ru,
URL: <http://borisovdi.narod.ru>

Abstract

We consider singular perturbed eigenvalue problem for Laplace operator in a cylinder with frequent and nonperiodic alternation of boundary conditions imposed on narrow strips lying in the lateral surface. The width of strips depends on a small parameter in an arbitrary way and may oscillate fast, moreover, the nature of oscillation is arbitrary, too. We obtain two-sided estimates for degree of convergences of the perturbed eigenvalues.

1 Introduction

This paper is devoted to the study of eigenvalue problem with frequent alternation of boundary conditions. The main feature of such boundary conditions is as follows. One should define the subset of a boundary consisting of a great number of disjoint closely-spaced parts of small measure. The boundary condition of one type is imposed on this subset, while one of another kind is settled on the rest part of the boundary. Papers devoted to homogenization of the elliptic problem with frequent alternation of boundary condition appeared in the second half of the last century (see, for instance, [8, 9, 10, 12, 16, 19]). Nonlinear elliptic boundary value problems involving frequent alternation of boundary conditions were treated in [1, 13]. In [1] they solved the problem numerically, while in [13] the homogenization theorems were proved. In the papers [14, 15] initial-boundary parabolic problem

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with frequent alternation of boundary condition were considered and, in particular, homogenization theorems were proved.

Asymptotics expansion for the solutions to the elliptic problems with frequently alternating boundary condition are constructed last decade. Two-dimensional case for periodic structure of alternation was studied well enough (see [6, 7, 17, 18] and References of these works), nonperiodic case was considered in [5]. Asymptotics expansions for three-dimensional problems with periodic structure of alternation were established in [2, 3, 4]. The leading terms of asymptotics expansions for the solution to a parabolic problem with frequent alternation of boundary conditions were obtained in [14, 15].

In the present paper we study eigenvalue problem for Laplace operator in a cylinder. The lateral surface is partitioned into great number of narrow strips on those the Dirichlet and Neumann condition are imposed in turns. We deal with essentially nonperiodic alternation, moreover, the width of the strips may varies arbitrarily, including the cases of slow and fast oscillations of complicated nature. Under minimal restrictions to the structure of alternations we give best possible two-sided estimates for degree of convergences of the perturbed eigenvalues.

We also note that estimates for degrees of convergence for another structure of alternation were obtained in [5, 11, 14, 15, 19], where parts of the boundary with different boundary condition were assumed to shrink to points.

2 Formulation of the problem and the main results

Let $x = (x', x_3)$, $x' = (x_1, x_2)$ be Cartesian coordinates, ω be an arbitrary bounded simply-connected domain in \mathbb{R}^2 having infinitely differentiable boundary, $\Omega = \omega \times [0, H]$ be a cylinder of height $H > 0$ with upper and lower basis ω_1 and ω_2 respectively. By s we denote the natural parameter of the curve $\partial\omega$, while ε is a small positive parameter: $\varepsilon = H/(\pi N)$, where $N \gg 1$ is a great integer. We define a subset of the lateral surface Σ of the cylinder Ω , consisting of N narrow strips (cf. figure):

$$\gamma_\varepsilon = \{x : x' \in \partial\omega, -\varepsilon a_j(s, \varepsilon) < x_3 - \varepsilon\pi(j + 1/2) < \varepsilon b_j(s, \varepsilon), j = 0, \dots, N - 1\}.$$

Here $a_j(s, \varepsilon)$ and $b_j(s, \varepsilon)$ are arbitrary functions belonging to $C^\infty(\partial\omega)$ and satisfying uniform on ε and j estimates:

$$0 < a_j(s, \varepsilon) < \frac{\pi}{2}, \quad 0 < b_j(s, \varepsilon) < \frac{\pi}{2}. \quad (1)$$

From geometrical point of view these estimates means that strips of the subset γ_ε associated with different values of j do not intersect.

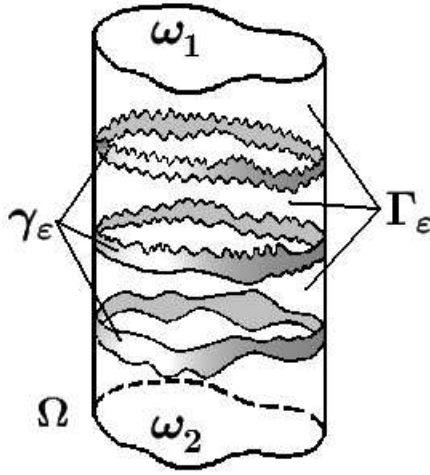


Figure 1:

We study singularly perturbed eigenvalue problem:

$$-\Delta\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon, \quad x \in \Omega, \quad (2)$$

$$\psi_\varepsilon = 0, \quad x \in \omega_1 \cup \gamma_\varepsilon, \quad \frac{\partial}{\partial\nu}\psi_\varepsilon = 0, \quad x \in \omega_2 \cup (\Sigma \setminus \overline{\gamma}_\varepsilon), \quad (3)$$

where ν is the outward normal.

The object of the paper is to find out the limits to the eigenvalues λ_ε and to estimate the degree of convergences as $\varepsilon \rightarrow 0$ under minimal restrictions for the set γ_ε . Earlier elliptic problem with boundary conditions (3) were studied in [2, 3, 4, 9]. In the paper [9] they studied the homogenization of the elliptic problem in a circular cylinder with boundary conditions (3) under additional assumption that $a_j(s, \varepsilon) = b_j(s, \varepsilon) = \eta(\varepsilon)$, $j = 0, \dots, N - 1$, where $\eta(\varepsilon)$ is a some function. Geometrically this assumption means that the set γ_ε is periodic and all strips have same constant width. In [3, 4] under the same assumptions complete asymptotics expansions for the problem (2), (3) were constructed. The case of arbitrary section and periodic distribution of the strips of slowly varying width was treated in [2, 3]. In [2, 3] it was assumed that $a_j(s, \varepsilon) = b_j(s, \varepsilon) = \eta(\varepsilon)g(s)$, where $g(s) \in C^\infty(\partial\omega)$, $0 < c_0 \leq g(s) \leq 1$, and the function $\eta(\varepsilon)$ satisfies the estimate $0 < \eta(\varepsilon) < \pi/2$. The leading terms of the asymptotics expansions for the eigenelements were constructed, moreover, the definition of a_j and b_j mentioned above played essential role in construction of these asymptotics. At the same time, the question on behaviour of the eigenvalues of the problem (2), (3) for arbitrary functions a_j and b_j is open. Clear, the arbitrary choice of the functions a_j and b_j leads, generally saying, to the set γ_ε of nonperiodic structure. Moreover, the functions $a_j(s, \varepsilon)$ and $b_j(s, \varepsilon)$ may oscillate fast on s . Such oscillation takes place under, for instance, following choice of the functions a_j and b_j : $a_j(s, \varepsilon) = b_j(s, \varepsilon) = \eta(\varepsilon)g(s/\alpha(\varepsilon))$, where $\eta(\varepsilon)$ is a rough width of the strip, $g \in C^\infty(\partial\omega)$, $\alpha(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$. We stress,

that in the example shown the leading terms of the asymptotics expansions for the eigenvalues of the problem (2), (3) will differ from the similar results of the papers [2, 3, 4] and will depend essentially on the function $\alpha(\varepsilon)$. If the set γ_ε is nonperiodic and the width of all or some strips oscillates fast, and also, these oscillations have arbitrary nonperiodic on s structure (cf. figure), the question of constructing the asymptotics expansions for the eigenvalues of the problem (2), (3) becomes very complicated and can be solved only under some additional constraints for the functions a_j and b_j . That's why to estimate the degree of convergence for the eigenvalues of the (2), (3) is a topical question. Exactly this question is solved in the present paper.

Now we proceed to the formulation of the main results. We take the perturbed eigenvalues in ascending order counting multiplicity: $\lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \dots \lambda_\varepsilon^k \leq \dots$. Limiting eigenvalues defined below are assumed to be taken in the same order: $\lambda_0^1 \leq \lambda_0^2 \leq \dots \lambda_0^k \leq \dots$.

Theorem 2.1. *Suppose the inequalities (1) and*

$$\eta(\varepsilon) \leq a_j(s, \varepsilon), \quad \eta(\varepsilon) \leq b_j(s, \varepsilon),$$

hold, where $\eta(\varepsilon)$ is an arbitrary function obeying the estimate $0 < \eta(\varepsilon) < \pi/2$ and the equality:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \ln \eta(\varepsilon) = 0.$$

Then uniform on ε and η estimates

$$-c_k \varepsilon (|\ln \eta| + 1) \leq \lambda_\varepsilon^k - \lambda_0^k \leq 0 \quad (4)$$

hold true. Here $c_k > 0$ are constants, λ_0^k are eigenvalues of the problem

$$\begin{aligned} -\Delta \psi_0 &= \lambda_0 \psi_0, \quad x \in \Omega, \\ \psi_0 &= 0, \quad x \in \omega_1 \cup \Sigma, \quad \frac{\partial}{\partial \nu} \psi_0 = 0, \quad x \in \omega_2. \end{aligned} \quad (5)$$

Theorem 2.2. *Suppose the inequalities*

$$\eta_0(\varepsilon) \eta(\varepsilon) \leq a_j(s, \varepsilon) \leq \eta(\varepsilon), \quad \eta_0(\varepsilon) \eta(\varepsilon) \leq b_j(s, \varepsilon) \leq \eta(\varepsilon),$$

holds, where $\eta_0(\varepsilon)$ and $\eta(\varepsilon)$ are arbitrary functions obeying the equalities:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \ln \eta_0(\varepsilon) = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \ln \eta(\varepsilon)} = -A, \quad A = \text{const} > 0.$$

Then uniform on ε and η estimates

$$-c_k (\mu(\varepsilon) + \varepsilon^2 + \varepsilon \ln \eta_0(\varepsilon)) \leq \lambda_\varepsilon^k - \lambda_0^k \leq c_k (\mu(\varepsilon) + \varepsilon^2) \quad (6)$$

hold true. Here $c_k > 0$ are constants,

$$\mu = \mu(\varepsilon) = -A - \frac{1}{\varepsilon \ln \eta(\varepsilon)},$$

λ_0^k are eigenvalues of the problem

$$\begin{aligned} -\Delta\psi_0 &= \lambda_0\psi_0, & x \in \Omega, & \quad \psi_0 = 0, & \quad x \in \omega_1, \\ \frac{\partial}{\partial\nu}\psi_0 &= 0, & x \in \omega_2, & \quad \left(\frac{\partial}{\partial\nu} + A\right)\psi_0 = 0, & \quad x \in \Sigma. \end{aligned} \quad (7)$$

Theorem 2.3. *Suppose the inequalities*

$$0 < a_j(s, \varepsilon) \leq \eta(\varepsilon), \quad 0 < b_j(s, \varepsilon) \leq \eta(\varepsilon)$$

hold, where $\eta(\varepsilon)$ is an arbitrary function obeying an equality:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \ln \eta(\varepsilon)} = 0.$$

Then uniform on ε and η estimates

$$0 \leq \lambda_\varepsilon^k - \lambda_0^k \leq c_k \mu(\varepsilon) \quad (8)$$

hold true. Here $c_k > 0$ are constants,

$$\mu = \mu(\varepsilon) = -\frac{1}{\varepsilon \ln \eta(\varepsilon)},$$

λ_0^k are eigenvalue of the problem (7) as $A = 0$.

Remark 2.1. We note that the hypotheses of the Theorems 2.1-2.3 admit the non-periodic set γ_ε . Moreover, this set may contain strips of width varying arbitrary, in particular, strips of fast oscillating width. For instance, the case shown in the figure is included under consideration.

Remark 2.2. We stress that the estimates (4), (6), (8) are best possible. More concretely, under hypothesis of each theorem there exist function a_j and b_j , for those the degree of convergence has the smallness order exactly as given in the estimates (4), (6), (8). This statement will be established in the proofs of Theorems 2.1-2.3.

3 Proof of Theorems 2.1-2.3

The proof of Theorems 2.1-2.3 is based on the following auxiliary statement.

Lemma 3.1. *Suppose the subsets $\gamma_{\varepsilon,*}$ and γ_ε^* of the lateral surface Σ satisfy*

$$\gamma_{\varepsilon,*} \subseteq \gamma_\varepsilon \subseteq \gamma_\varepsilon^*, \quad (9)$$

and let $\lambda_{\varepsilon,}^k$ and $\lambda_\varepsilon^{*,k}$ be eigenvalues of the problem (2), (3) with γ_ε replaced by $\gamma_{\varepsilon,*}$ and γ_ε^* , respectively. Then the estimates*

$$\lambda_{\varepsilon,*}^k \leq \lambda_\varepsilon^k \leq \lambda_\varepsilon^{*,k} \quad (10)$$

are valid.

This lemma can be proved in a standard way on the base of minimax property of the eigenvalues of the problem (2), (3).

Proof of Theorem 2.1. Let

$$\gamma_{\varepsilon,*} = \{x : x' \in \partial\omega, |x_3 - \varepsilon\pi(j + 1/2)| < \varepsilon\eta(\varepsilon), j = 0, \dots, N-1\}, \quad \gamma_{\varepsilon}^* = \Sigma.$$

In accordance with hypothesis, these sets meet the inclusions (9). The set $\gamma_{\varepsilon,*}$ being periodic, the asymptotics

$$\lambda_{\varepsilon,*}^k = \lambda_0^k + \varepsilon\lambda_1^k \ln \sin \eta + o(\varepsilon(|\ln \eta| + 1)), \quad (11)$$

hold. Here λ_1^k are some constants, λ_0^k are eigenvalues of the problem (5). In the case of simple eigenvalue λ_0^k this asymptotics was constructed formally in [2]. The case of eigenvalue λ_0^k of arbitrary multiplicity was studied rigorously in [4] under assumption $\eta = \text{const}$, ω is a unit circle. These additional assumptions are not essential and the asymptotics (11) can be proved completely by analogy with [2, 4]; the rigorous two-parametrical estimates for the error term can be easily established on the base of the results of [6]. Since $\gamma_{\varepsilon}^* = \Sigma$, it follows that $\lambda_{\varepsilon}^{*,k} = \lambda_0^k$. Substituting these equalities and asymptotics (11) into (10), we get

$$\lambda_0^k + c_k \varepsilon(|\ln \eta| + 1) \leq \lambda_{\varepsilon}^k \leq \lambda_0^k,$$

where $c_k > 0$ are some constants. This implies the needed two-sided estimates. These estimates are best possible. Indeed, taking $a_j(s, \varepsilon) = b_j(s, \varepsilon) = \eta(\varepsilon)$, we see that λ_{ε}^k meet asymptotics (11), i.e.,

$$\lambda_{\varepsilon}^k - \lambda_0^k = O(\varepsilon(|\ln \eta| + 1)).$$

□

Proof of Theorem 2.2. The scheme of the proof is same with one of Theorem 2.1. The sets $\gamma_{\varepsilon,*}$ and γ_{ε}^* should be defined as follows

$$\begin{aligned} \gamma_{\varepsilon,*} &= \{x : x' \in \partial\omega, |x_3 - \varepsilon\pi(j + 1/2)| < \varepsilon\eta_0(\varepsilon)\eta(\varepsilon), j = 0, \dots, N-1\}, \\ \gamma_{\varepsilon}^* &= \{x : x' \in \partial\omega, |x_3 - \varepsilon\pi(j + 1/2)| < \varepsilon\eta(\varepsilon), j = 0, \dots, N-1\}. \end{aligned}$$

By the hypothesis, these sets obey inclusions (9). The corresponding eigenvalues $\lambda_{\varepsilon,*}^k$ and $\lambda_{\varepsilon}^{*,k}$ have the asymptotics

$$\begin{aligned} \lambda_{\varepsilon,*}^k &= \lambda_0^k + \tilde{\mu}\lambda_{0,1}^k + \varepsilon^2\lambda_{1,0}^k + o(|\tilde{\mu}| + \varepsilon^2), \\ \lambda_{\varepsilon}^{*,k} &= \lambda_0^k + \mu\lambda_{0,1}^k + \varepsilon^2\lambda_{1,0}^k + o(|\mu| + \varepsilon^2). \end{aligned}$$

Here $\lambda_{i,j}^k$ are some constants, λ_0^k are eigenvalues of the problem (7),

$$\tilde{\mu} = \tilde{\mu}(\varepsilon) = -A - \frac{1}{\varepsilon \ln \eta_0 \eta}.$$

For the case of simple eigenvalue λ_0^k and arbitrary section ω , as well as for the case of multiply eigenvalue λ_0^k and circular section ω these asymptotics were found in [2, 3]. The technique of these two papers can be easily applied to the case of arbitrary section ω and multiply eigenvalue λ_0^k . The asymptotics for $\lambda_{\varepsilon,*}^k$ and $\lambda_{\varepsilon}^{*,k}$, definition of $\tilde{\mu}$ and inequalities (10) yield needed estimates for degree of convergence. Unimprovability for these estimates is established by analogy with Theorem 2.1. \square

The proof of Theorem 2.3 is similar to proof of Theorems 2.1, 2.2. Here sets $\gamma_{\varepsilon,*}$ and γ_{ε}^* and asymptotics for corresponding eigenvalues $\lambda_{\varepsilon,*}^k$, $\lambda_{\varepsilon}^{*,k}$ ([2, 3]) are as follows:

$$\begin{aligned}\gamma_{\varepsilon,*} &= \emptyset, & \gamma_{\varepsilon}^* &= \{x : x' \in \partial\omega, |x_3 - \varepsilon\pi(j + 1/2)| < \varepsilon\eta(\varepsilon), j = 0, \dots, N-1\}, \\ \lambda_{\varepsilon,*}^k &= \lambda_0^k, & \lambda_{\varepsilon}^{*,k} &= \lambda_0^k + \lambda_{0,1}^k \mu + o(\mu),\end{aligned}$$

where $\lambda_{0,1}^k$ are some constants, λ_0^k are eigenvalues of the problem (7) with $A = 0$.

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